The environment, the endogenous growth and the endogenous labour market

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Abstract

In the framework of endogenous growth of Acemoglu & Aghion in " The Environment and Directed Technical Change", the paper aims at creating an endogenous labor market and analyzing its effects on the environment and on the growth in the competitive equilibrium and the optimal policy. A final output is produced by two intermediary goods, one is "dirty" and the other is "clean". The "dirty" one lowers the quality of the environment. On the contrary to the initial model, we figure out the long term growth rate changes in both the competitive equilibrium and the optimal policy. As usual in the macroeconomics models with micro foundations, we find a threshold for the discount factor ρ^* when the asymptotic growth rate in competitive equilibrium is higher than in the optimal policy. Moreover, on the contrary to the initial paper, there is a case in which an environmental disaster can be avoided in competitive equilibrium when the expected enhanced productivity is not sufficiently high. Hence, there is no growth but also no disaster. In addition to the temporary policy instruments used in the initial paper, the social planner needs another subsidy which has to be permanent for incenting the workers to become scientists in the clean sector. Hence, on the contrary to the initial paper, the government intervention has to be permanent. Therefore, our results are less optimistic than those of Acemoglu & Aghion. Our results are close to Stern/ Al Gore stance. Lastly, we find that if ρ is sufficiently high, there is no asymptotic growth in the optimal policy as in the competitive equilibrium and therefore no possible degradation of the environment.

1 The framework and the main results

In this section, we remind and simplify as much as possible the initial framework of the paper. We explain and demonstrate the main results used in the initial paper and necessary for our analysis. The proofs are shown in the appendix.

The economy admits a representative household with preferences:

$$\sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} u(C_t, S_{t)} \tag{1}$$

where C_t is the consumption of the final good at time t, S_t represents the quality of the environment at time t and ρ is the discount rate. We assume $S_t \in \left[0, \overline{S}\right]$, where $\overline{S}$ is the highest possible quality of the environment, and $S_0 = \overline{S}$, and 0 the lowest. Hence, S_t is bounded.

$u(C_t, S_{t)}$ is increasing and concave in S_t and C_t and twice differentiable in both its arguments.

Moreover, $u(C, S)$ verifies the usual Inada conditions:

$$\lim_{C \to 0} \frac{\partial u(C,S)}{\partial C} = \infty, \ \lim_{S \to 0} \frac{\partial u(C,S)}{\partial S} = \infty, \ \lim_{S \to 0} u(C,S) = -\infty \tag{2}$$

and

$$\frac{\partial u(C, \overline{S})}{\partial S} = 0 \tag{3}$$

which means that as soon as $\overline{S}$ is reached, an improvement of the environment will not increase the utility.

The final good Y_t is produced competitively thanks to a CES function using "clean" Y_{ct} and "dirty" Y_{dt} inputs:

$$Y_t = (Y_{ct}^{\frac{\varepsilon-1}{\varepsilon}} + Y_{dt}^{\frac{\varepsilon-1}{\varepsilon}})^{\frac{\varepsilon}{\varepsilon-1}} \tag{4}$$

where ε is the elasticity of substitution, we assume for the following that $\varepsilon > 1$,meaning that the clean sector is able to replace the dirty sector in the production of the final good.

The "clean" Y_{ct} and "dirty" Y_{dt} inputs are produced competitively (price takers) thanks to Cobb-Douglas functions:

$$Y_{ct} = L_{ct}^{1-\alpha} \int_0^1 A_{cit}^{1-\alpha} x_{cit}^{\alpha} di \tag{5}$$

and

$$Y_{dt} = L_{dt}^{1-\alpha} \int_0^1 A_{dit}^{1-\alpha} x_{dit}^{\alpha} di \tag{6}$$

where $\alpha \in [0,1]$, L_{dt} and L_{ct} stand for the proportions of workers in the dirty and the clean sectors.

In the initial paper, the total population of workers L_t is settled to 1 (living one period):

$$L_{ct} + L_{dt} \leq 1 \tag{7}$$

A_{dit} and A_{cit} represent the productivities of each intermediate machine i in the dirty and the clean sectors. Following the previous literature on the endogenous technical change, the intermediate machines x_{jit} are produced monopolistically by competitive firms and suffer from total depreciation at each period. We set the cost of producing one intermediate machine to α^2, which represents the investment in a machine i in sector j. We are in a full depreciation scheme for the intermerdiary machines x_{jit}.

The market clearing equation for the final good is:

$$C_t = Y_t - \alpha^2 \left(\int_0^1 x_{cit} di + \int_0^1 x_{dit} di \right) \tag{8}$$

At the beginning of each period, each scientist chooses the clean or dirty sectors given its potential expected profits. After, the scientist is randomly allocated to one of the machine in the sector and discovers an innovation with a probability η_d if he works on a machine of the dirty sector and a probability η_c if he works on a machine of the clean sector. In both cases, the improvement of productivity of the machine is given by γ. Hence, A_{jit} the productivity of the machine i in the sector $j = c, d$ at time t, is a geometrical progression:

$$A_{jit} = (1 + \gamma \eta_j 1_{s_{jt}}(i)) A_{jit-1} \tag{9}$$

where A_{jit} the productivity of the machine i in the sector $j = c, d$ at time $t-1$, and $1_{s_{jt}}(i)$ is the indicative function equal to 1 if the scientist is allocated to the machine i in sector j and 0 either.

Then, if successful, the scientist obtains a patent for the improved machine and hence holds a monopoly of the machine in the current period. In the initial paper, the total population of scientist is settled to 1 (living one period):

$$s_{ct} + s_{dt} \leq 1 \tag{10}$$

Then, we define

$$A_{dt} = \int_0^1 A_{dit} di \text{ and } A_{ct} = \int_0^1 A_{cit} di \tag{11}$$

as the average productivities of the dirty and the clean sectors.

Hence, by summing (9) for each i for $j = c, d$:

$$A_{dt} = (1 + \gamma \eta_d s_{dt}) A_{dt-1} \text{ and } A_{ct} = (1 + \gamma \eta_c s_{ct}) A_{ct-1} \tag{12}$$

The evolution of the environment is given by

$$S_{t+1} = -\xi Y_{dt} + (1 + \delta) S_t \tag{13}$$

where ξ represents the degradation of the environment: the more the dirty input is used, the more the quality of the environment is lowered and δ stands for the capacity of regeneration of the environment.

Definition 1 *When $S_t = 0$, for $t < \infty$, an environmental disaster happens. Reversibility is impossible:*
if $\exists\ t_d$ such that $S_{t_d} = \overline{S}, \forall t > t_d$, $S_t = 0$.

2 The competitive equilibrium and the labour market

In this section, we focus on the competitive equilibrium. The economy is decentralized, there is no government intervention.

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2.1 The competitive equilibrium in the initial framework

In this subsection, we follow the initial paper and simplify as much as possible the results.

Definition 2 *An equilibrium at time t is given by sequences of wages (w_t), prices for inputs (p_{jt}), prices for machines (p_{jit}), demands for machines (x_{jit}), demands for inputs (Y_{jt}), labor demands (L_{jt}) by input producers $j \in \{c, d\}$, scientists proportions allocations (s_{ct},s_{dt}) and quality of environment (S_t) such that: (i) (p_{jit}; x_{jit}) maximizes profit by the producer of machine in sector j; (ii) Ljt maximizes profits by producers of input j; (iii) Y_{jt} maximizes the good producers; (iv) (s_{ct},s_{dt}) maximizes the expected profit of a researcher at date t; (v) the wage w_t and the prices p_{jt} clear the labor and input markets respectively; and (vi) the evolution of S_t is given by (13)*

Assumption 1 We assume, at date $t = 0$ that the dirty sector is sufficiently more advanced than the clean sector so that scientists choose the dirty sector at date $t = 0$. It occurs if and only if:

$$\frac{A_{c0}}{A_{d0}} < \min\left\{(1+\gamma\eta_c)^{-\frac{\varphi+1}{\varphi}}\left(\frac{\eta_c}{\eta_d}\right)^{\frac{1}{\varphi}}, (1+\gamma\eta_d)^{\frac{\varphi+1}{\varphi}}\left(\frac{\eta_c}{\eta_d}\right)^{\frac{1}{\varphi}}\right\}$$

Due to the "building on the shoulder's giant scheme" occurring in this model of endogenous growth, if there is no intervention, it is sufficient to have the dirty sector more advanced than the clean sector at date $t = 0$ for having it for $t > 0$.

We demonstrate the following results in the appendix.

From the fact that the final good Y_t is produced competitively we have,

$$\frac{p_{ct}}{p_{dt}} = \left(\frac{Y_{ct}}{Y_{dt}}\right)^{-\frac{1}{\varepsilon}} \tag{14}$$

For each period t, we normalize the price of the final good to 1. Hence,

$$(p_{ct}^{1-\varepsilon} + p_{dt}^{1-\varepsilon})^{\frac{1}{1-\varepsilon}} = 1$$

The profit of a monopolist producer of the machine i in sector j is

$$\pi_{jit} = (1-\alpha)\alpha p_{jt}^{\frac{1}{1-\alpha}} L_{jt} A_{jit}$$

and the expected profit of a scientist who decides to work in the sector j is

$$\Pi_{jt} = \eta_j(1+\gamma)(1-\alpha)\alpha p_{jt}^{\frac{1}{1-\alpha}} L_{jt} A_{jt-1} \tag{15}$$

Consequently, the allocation of the scientists is driven by the following ratio:

$$\frac{\Pi_{ct}}{\Pi_{dt}} = \frac{\eta_c}{\eta_d} \times \frac{L_{ct}}{L_{dt}} \times \left(\frac{p_{ct}}{p_{dt}}\right)^{\frac{1}{1-\alpha}} \times \frac{A_{ct-1}}{A_{dt-1}}$$

Then, the allocation of scientists is lead by three effects: the R&D, the market size and the price effects.

Thanks to (40), (41), we obtain:

$$\frac{\Pi_{ct}}{\Pi_{dt}} = \frac{\eta_c}{\eta_d} \times \left(\frac{1+\gamma\eta_c s_{ct}}{1+\gamma\eta_d s_{dt}}\right)^{-\varphi-1} \times \left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-\varphi} \tag{16}$$

Finally, taking the previous equations, we obtain the expressions of Y_{dt}, Y_{ct} and Y_t with the average productivity A_{dt} and A_{ct}.

We set $\varphi = (1-\alpha)(1-\varepsilon) < 0$.

$$\begin{aligned} Y_{dt} &= (A_{ct}^{\varphi} + A_{dt}^{\varphi})^{-\frac{\alpha+\varphi}{\varphi}} A_{ct}^{\varphi+\alpha} A_{dt},\ Y_{ct} = (A_{ct}^{\varphi} + A_{dt}^{\varphi})^{-\frac{\alpha+\varphi}{\varphi}} A_{dt}^{\varphi+\alpha} A_{ct} \quad (17) \\ Y_t &= (A_{ct}^{\varphi} + A_{dt}^{\varphi})^{-\frac{1}{\varphi}} A_{ct} A_{dt} \end{aligned}$$

(17) is central since it expresses the dirty and clean sectors inputs and the final output in function of the average productivities. The average productivities are improved thanks to the endogenous scheme, then we can deduce the growth of the dirty and clean inputs, and the one of the economy (the final output)

Proposition 3 *We suppose that assumption 1 is verified, then there is a unique equilibrium where innovation occurs always in the dirty sector only and the long term growth of the dirty input and the long term growth of the economy are both equal to* $\gamma\eta_d$.

Moreover, since the environmental stock is bounded by $\overline{S}$, once reaches 0 there is no possible backward step and the dirty sector grows asymptotically at the rate $\gamma\eta_d$, there is a date t_d, when $S_{t_d} = 0$.

Theorem 4 *There will be an environmental disaster if assumption 1 is verified and if there is no exogenous intervention.*

As in the previous litterature on endogenous growth, the economic growth is affected by the growth of "scientific progress" expressed by (12). Since all the scientists go to the dirty sector (proportion equal to 1), the economic growth is equal to $\gamma\eta_d$. ($\gamma\eta_d \times 1$)

2.2 The competitive equilibrium with an endogenous labour market

The idea is to allow scientists to become workers and vice versa. Here, not only do we allow the scientists to work in the sector in which they have the higher expected profit, but we also allow them to become workers and vice versa. Consequently, there is *full mobility across the sectors and the jobs.* By releasing the constraints of prohibiting job mobility (10) and (7), we expect genuine modifications of the initial model.

If assumption 1 holds, we do think that the growth of the dirty input can be altered since on the one hand scientists heading towards the dirty sector may be attracted by the job of the workers in both sectors, and the growth rate of the dirty sector could be lowered. But on the other hand, the workers can choose to work as scientists in the dirty sector and hence increasing the growth rate of the dirty sector. Hence, the date of the environmental disaster may be modified.

We constraint the total population to one

$$s_{ct} + s_{dt} + L_{dt} + L_{ct} \leq 1 \tag{18}$$

NB: We constraint total population to 1 in order to match with the number of i machines (continium equal to 1). If total population was greater than 1, there would be problems of sharing profits if a number superior to 1 of scientists succeeded, and there would not be monopoly on the machines modifying all the computations.

According to the previous computations and because of mobility across the clean and dirty sectors and no " labour externality ", the wages in both sectors must be equal at equilibrium. The wage of a worker at time t in sector j is given by (38) and the expected profit of a scientist at time t in sector j is given (15).

Theorem 5 *The equilibrium is reached in the labour market when:*

- the wage of a worker in the dirty sector is equal to the wage of a worker in the clean sector

- the maximum expected profit of a scientist among the clean and the dirty sectors is equal to the wage of a worker.

In order to have the equilibrium in the labour market, we equalize the maximum expected profit of a scientist with the wage of a worker. We equalize an expected income with a permanent one, which means that agents are risk neutral. However, we remind that since u is concave, agents are risk averse. In order to make this coherent, we find that introducing a financial market of Arrow securities allows the scientists insuring against " the no discovering case" for the period they live. At a micro level, as proved in finance, they only have to equalize the expected profits, which is exactly what we do. At a macro level, nothing is changed in comparison to the case without a financial market, since we dealt with aggregated output and consumption (the market clearing "erases" the effect of the financial market and we are in perfect information and the financial market is costless).

NB: The paper dealts with aggregated variables. However, we remind that introducing a financial market will improve the allocations of each individual intertemporally according to Arrow results.

Consequently, the equilibrium in the labour market will be given by the following system:

$$(1-\alpha)\, p_{dt} L_{dt}^{-\alpha} \int_0^1 A_{dit}^{1-\alpha} x_{dit}^{\alpha} di = \max_{j=c,d} \eta_j (1+\gamma)(1-\alpha)\alpha p_{jt}^{\frac{1}{1-\alpha}} L_{jt} A_{jt-1}$$

$$(1-\alpha)\, p_{ct} L_{ct}^{-\alpha} \int_0^1 A_{cit}^{1-\alpha} x_{cit}^{\alpha} di = (1-\alpha)\, p_{dt} L_{dt}^{-\alpha} \int_0^1 A_{dit}^{1-\alpha} x_{dit}^{\alpha} di$$

$$\text{which yields } \frac{L_{ct}}{L_{dt}} = \left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi}$$

$$s_{ct} + s_{dt} + L_{dt} + L_{ct} = 1$$

$$s_{dt}, s_{ct}, L_{dt}, L_{ct} \geq 0$$

However, as in the initial paper, we assume the dirty sector is initially more advanced than the clean sector. Consequently, due to the " building on the giant's shoulders scheme ", the expected profit will be higher in the dirty sector than in the clean sector forever if there is no intervention if assumption 1 is verified.

PS: On the contrary to the initial model, assumption 1 is only sufficient but not necessary for having the dirty sector more advanced than the clean sector.

Hence, the system becomes:

$$(1-\alpha)\, p_{dt} L_{dt}^{-\alpha} \int_0^1 A_{dit}^{1-\alpha} x_{dit}^{\alpha} di = \ \eta_d (1+\gamma)(1-\alpha)\alpha p_{dt}^{\frac{1}{1-\alpha}} L_{dt} A_{dt-1}$$

$$(1-\alpha)\, p_{ct} L_{ct}^{-\alpha} \int_0^1 A_{cit}^{1-\alpha} x_{cit}^{\alpha} di = (1-\alpha)\, p_{dt} L_{dt}^{-\alpha} \int_0^1 A_{dit}^{1-\alpha} x_{dit}^{\alpha} di$$

which yields $\frac{L_{ct}}{L_{dt}} = \left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi}$

$$s_{ct} + s_{dt} + L_{dt} + L_{ct} = 1$$

$$s_{ct} = 0$$

which yields $s_{dt} + L_{dt} + L_{ct} = 1$

$$s_{dt}, L_{dt}, L_{ct} \geq 0$$

We solve the system and we have the proportions of workers and scientists at equilibrium at time t.

The computations (19), (20), (21) are given in the appendix.

- if $\alpha\eta_d(1+\gamma) - 1 - \left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi} \geq 0$:

$$s_{dt} = \frac{1 - \frac{1}{\alpha\eta_d(1+\gamma)}\left(\left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi} + 1\right)}{1 + \frac{\gamma\eta_d}{\alpha\eta_d(1+\gamma)}\left(\left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi} + 1\right)} \tag{19}$$

$$L_{dt} = \frac{1 + \eta_d \gamma \frac{1 - \frac{1}{\alpha\eta_d(1+\gamma)}\left(\left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi}+1\right)}{1 + \frac{\gamma\eta_d}{\alpha\eta_d(1+\gamma)}\left(\left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi}+1\right)}}{\alpha\eta_d\,(1+\gamma)} \tag{20}$$

$$L_{ct} = \left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi}\left(\frac{1 + \eta_d \gamma \frac{1 - \frac{1}{\alpha\eta_d(1+\gamma)}\left(\left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi}+1\right)}{1 + \frac{\gamma\eta_d}{\alpha\eta_d(1+\gamma)}\left(\left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi}+1\right)}}{\alpha\eta_d\,(1+\gamma)}\right) \tag{21}$$

- $\alpha\eta_d(1+\gamma) - 1 - \left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi} < 0$

$$s_{dt} = 0$$

$$L_{dt} = \frac{1}{1 + (\frac{A_{ct}}{A_{dt}})^{-\varphi}}$$

$$L_{ct} = \frac{(\frac{A_{ct}}{A_{dt}})^{-\varphi}}{1 + (\frac{A_{ct}}{A_{dt}})^{-\varphi}}$$

We go back to the case $\alpha\eta_d(1+\gamma)-1-\left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi} \geq 0$ analyzing the asymptotic results.

Taking the limit, we deduce the asymptotic proportion of scientists s_d. Since A_{dt} grows exponentially, A_{ct} remains constant and $\varphi < 0$: $\left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi} \to 0$

$$s_d = \lim_{t\to\infty} s_{dt} = \frac{\alpha\eta_d(1+\gamma)-1}{\alpha\eta_d(1+\gamma)+\gamma\eta_d} \tag{22}$$

$$L_d = \frac{\gamma\eta_d + 1}{\alpha\eta_d(1+\gamma)+\gamma\eta_d}$$

$$L_c = 0$$

Hence if $\alpha\eta_d(1+\gamma)-1 > 0$, the asymptotic proportion of scientists working the dirty sector (more advanced than the clean sector) will be stricty positive, consequently there will be endogenous growth.

Either, the asymptotic proportion of scientists working in the dirty sector will be equal to 0.

Conclusion 6 *The proportion of scientists will be driven by three different parameters. First, the higher η_d and γ are, the higher the proportion of scientists is. Indeed, the probability of innovation and the growth of enhanced productivity represented by η_d and γ affect directly and positively the work of a scientist. Second, the higher α is, the higher the proportion of scientists is. The effect of α on the distribution of workers and scientists is more ambiguous since α affects both the wage and the expected profit of a scientist. However, through the role of productivity in the Cobb Douglas functions, α affects positively the proportion of scientists and therefore negatively the proportion of workers.*

2.3 The effects on the economy: environmental disaster and asymptotic growth rate

If $\alpha\eta_d(1+\gamma)-1 > 0$, we deduce the asymptotic growth rate g_d of the dirty sector, which grows exponentially, thanks to (17):

$$g_d = \gamma\eta_d s_d = \gamma\eta_d\left(\frac{\alpha\eta_d(1+\gamma)-1}{\alpha\eta_d(1+\gamma)+\gamma\eta_d}\right)$$

Theorem 7 *Taking into account the equation of the evolution of the environment stock, the fact that S_t is bounded by $\overline{S}$ and the growth of the dirty sector, there will be an environmental disaster.*

Thanks to (17), we deduce the growth rate of the economy g

$$g = g_d$$

On the contrary, if $\alpha\eta_d(1+\gamma)-1 \leq 0$, there is no scientist in the economy. Thanks to (17), we deduce the growth rate equal to 0. In this case, since there is no growth, given (8), consumption remains constant. However, the dirty input does not grow anymore, and because of the regeneration scheme of the environment (13), S_t grows and therefore the utility $u(\overline{C}, S_t)$ will grow until $u(\overline{C}, \overline{S})$, with $\overline{C}$ the constant consumption. Hence, it is a case of **sustainable development** without government intervention.

$$g = g_d = 0$$

The dynamic of S_t can be written as:

$$S_t = (1+\delta)^{t+1}(\overline{S} - \xi\frac{Y_{do}}{\delta}) + \xi\frac{Y_{do}}{\delta} \tag{23}$$

Even if the growth is equal to 0, due to the environment regeneration, the intertemporal utility will grow until $u(\overline{C}, \overline{S})$.(assuming that $\overline{S} - \xi\frac{Y_{do}}{\delta} > 0$)

Theorem 8 *If the product of the expected productivity by the Cobb-Douglas coefficient is inferior to 1, there will be no growth of the dirty input. Consequently, there will be no environmental disaster. But even if the growth is equal to 0, due to the environment regeneration, the intertemporal utility* $u(\overline{C}, S_t)$ *will grow until* $u(\overline{C}, \overline{S})$.

Actually, here lies a paradoxical result. Through the improvement of the productivity, the innovation should always create growth. Here, we find a critical threshold that the product of the expected productivity and the Cobb Douglas coefficient has to met in order to create growth. In fact, the improvement of the productivity is taken into account through the expected productivity and α. *This occurs because of total labour mobility across the sectors and the jobs.*

If the expected productivity is sufficiently high, as in the initial paper, the growth triggers an environmental disaster and lowers the utility to 0 (taking the Nordhaus utility function). This occurs because of potential negative externalities of the dirty sector and the fact that the dirty sector is more advanced than the clean sector.

Conclusion 9 *The results presented here can be compared only qualitatively to the ones of the initial paper. Indeed, in competitive equilibrium the authors found that there will be an environmental disaster if only if the dirty sector is more advanced than the clean sector. We find that in competitive equilibrium there will be an environmental disaster if the dirty sector is more advanced than the clean sector and if the product between the expected productivity and the Cobb Douglas coefficient is superior to one (hence the conditions for an environmental disaster are more restrictive). We find a paradoxical result: when there is no growth and no environmental disaster, the utility can still increase at short term because of environmental regeneration.*

PS: We can compare qualitatively and not quantitatively since we have modified the model by restricting the entire population to 1. In the initial paper, the whole population was set to 2.(scientists + workers). Hence, the asymptotic growth rates cannot be compared, but only the variables which affect it. As in the initial paper, the asymptotic growth rate is affected by the productivity and the probability of discovery. However, in our framework, through the equalization of incomes between the scientist and the worker, α is a new variable modifying positively the asymptotic growth rate.

2.4 Preventing the environmental disaster: redirecting the innovation

In the initial paper, the authors show that if a subsidy q_t is set such that:

$$\frac{\Pi_{ct}}{\Pi_{dt}} = \left(\frac{\eta_c}{\eta_d}\right)\left(\frac{1+\gamma\eta_c s_{ct}}{1+\gamma\eta_d s_{dt}}\right)^{-\varphi-1}\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-\varphi}(1+q_t) > 1$$

Then, the environmental disaster can be avoided. We remind that this is not yet the optimal policy which involves other instruments. A temporary subsidy is sufficient to redirect the innovation towards the clean sector. Since, the quantity of scientists in a sector drives the growth of the economy, the growth rate of the dirty sector will be equal to 0 and the disaster is avoided. From this, the authors compute the cost of the delayed intervention by equalizing the quantity of final output if the dirty sector is still used and the one when there is a shift towards the clean sector which is from (17):

$$Y_t = \left(A_{ct-1}^{\varphi} + A_{dt-1}^{\varphi}(1+\gamma\eta_d)\right)^{-\frac{1}{\varphi}} A_{ct-1}A_{dt-1}(1+\gamma\eta_d) = \left(A_{ct-1}^{\varphi}(1+\gamma\eta_c)^t + A_{dt-1}^{\varphi}\right)^{-\frac{1}{\varphi}} A_{ct-1}(1+\gamma\eta_c)^t A_{dt-1}$$

which yields

$$T_t = \frac{\ln((1+\gamma\eta_d)^{-\varphi}-1)(\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-\varphi}+1)}{-\varphi\ln(1+\gamma\eta_c)}$$

representing the number of periods necessary to reach the GDP if there would not have been any intervention (redirecting from the dirty towards the clean sector). It can be shown that $T_t \geq 2$, if $t \geq 1$. Hence, there is always a cost for delaying the intervention.

In our framework, the temporary intervention with the implementation q_t still holds and permits to avoid the environmental disaster. However, computing analytically the cost of delay is almost impossible. In our framework, there can be "comings and goings" between the two different jobs (scientist/ worker). Hence, on the contrary to the initial paper, it is impossible to set the total proportion of scientists to 1; this proportion can be altered. Consequently, we cannot compute the cost of delayed intervention.

3 Optimal policy and labour market

Given the existence of externalities, positive and negative, the optimal policy derived from the maximization of the program will be different from the competitive equilibrium of the previous section. We focus on the necessary fiscal instruments in order to make the competitive equilibrium matching with the optimal allocation of ressources.

3.1 The optimal policy in the initial paper

We try to simplify as much as possible the results found highlighting those which will be necessary for our own analysis. As usual, the optimal policy is obtained by maximizing the program which is:

$$\underset{Y_t,Y_{ct},Y_{dt},C_t,S_t,x_{dit},x_{cit},L_{dt},L_{ct},s_{ct},s_{dt},A_{ct},A_{dt},A_{dit},A_{cit}}{Max} \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} u(C_t, S_t)$$

under the constraints (4),(5),(6),(7),(8),(9),(10),(11),(12),(13).
In order to tackle the negative externality of Y_{dt}, we set a tax τ_t :

$$\tau_t = \frac{\xi \sum\limits_{v=t+1}^{\infty} (1+\delta)^{v-(t+1)} \frac{1}{(1+\rho)^{t+1}} \frac{\partial u(C_v,S_v)}{\partial S_v} 1_{S_{t+1},\ldots,S_v<\overline{S}}}{\lambda_{dt}} \tag{24}$$

Furthermore, we set a subvention equal to $1-\alpha$ for the negative externality of the monopoly of x_{jit} and consequently, the production of each input is changed and scaling by $\alpha^{-\frac{\alpha}{1-\alpha}}$ in comparison to the competitive equilibrium situation:

$$Y_{jt} = \left(\frac{1}{\alpha}\widehat{p_{jt}}\right)^{\frac{\alpha}{1-\alpha}} A_{jt} L_{jt}$$

The social planner will allocate the scientists to the clean sector whenever the ratio of the social value of a scientist in the clean sector and of the one's of a scientist in the dirty sector is superior to 1:

$$\gamma\eta_d(1+\gamma\eta_d s_{dt})^{-1} \sum_{\tau\geq t} \lambda_\tau \left(\frac{\lambda_{dt}}{\lambda_t}\right)^{\frac{1}{1-\alpha}} L_{d\tau} A_{d\tau} < \gamma\eta_c(1+\gamma\eta_c s_{ct})^{-1} \sum_{\tau\geq t} \lambda_\tau \left(\frac{\lambda_{ct}}{\lambda_t}\right)^{\frac{1}{1-\alpha}} L_{c\tau} A_{c\tau} \tag{25}$$

hence given the tax and the subvention, the new profit ratio which is:

$$\frac{\Pi_{ct}}{\Pi_{dt}} = \left(\frac{\eta_c}{\eta_d}\right) \left(\frac{1+\gamma\eta_c s_{ct}}{1+\gamma\eta_d s_{dt}}\right)^{-\varphi-1} \left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-\varphi} (1+\tau_t)^{\varepsilon}$$

has to be superior to 1 in order to attract scientists in the clean sector, achievable by setting a temporary subsidy q_t such that:

$$\frac{\Pi_{ct}}{\Pi_{dt}} = \left(\frac{\eta_c}{\eta_d}\right)\left(\frac{1+\gamma\eta_c s_{ct}}{1+\gamma\eta_d s_{dt}}\right)^{-\varphi-1}\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-\varphi}(1+\tau_t)^{\varepsilon}(1+q_t) > 1 \tag{26}$$

Conclusion 10 *In the initial paper, the optimal policy is temporary and composed by a subsidy tailored to redirect the scientists towards the clean sector, a tax for the dirty input.*

3.2 The optimal policy with an endogenous labor market

Creating an endogenous market implies modifying some equations of the initial paper. We introduce L_t as the total proportion of workers and s_t as the total proportion of scientists. The new constraints are:

$$s_{ct} + s_{dt} \leq s_t \tag{27}$$

$$L_{ct} + L_{dt} \leq L_t \tag{28}$$

$$L_t + s_t \leq 1 \tag{29}$$

By doing such thing, we do not alter the equations of the initial paper but we introduce new equations and we match with (18)

Since $\varepsilon > 1$, it is always optimal to produce both in the clean and in the dirty sectors . Hence, we do not have to take care of the slackness conditions for L_{dt} and L_{ct} :

$$L_{jt} > 0 \quad j = c, d$$

but the slackness conditions for s_{ct} and s_{dt} remain

$$\begin{aligned} s_{ct} &\geq 0 \\ s_{dt} &\geq 0 \end{aligned} \tag{30}$$

The new program is

$$\underset{Y_t, Y_{ct}, Y_{dt}, C_t, S_t, x_{dit}, x_{cit}, L_{dt}, L_{ct}, L_t, s_{ct}, s_{dt}, s_t, A_{dt}, A_{ct}}{Max} \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} u(C_t, S_{t)}$$

under the constraints (4),(5),(6),(8),(9),,(11),(12), (13) and the new constraints (27) ,(28) ,(29), (30):

We remind that the Lagrangean multiplier represents the shadow price or the social value of the variable.

The social planner will redirect the workers and the scientists across the sectors *and* the jobs thanks to subsidies and tax in order to make the order of the profits matching with the order set by the social values. Applying the usual welfare theorems, the social optimum will differ from the competitive equilibrium for the variables which create externalities (positive and negative).

According to the computations, whenever the social value of a scientist working in the clean sector is superior to the one of a scientist in the dirty sector:

$$\gamma\eta_d(1+\gamma\eta_d s_{dt})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{dt}}{\lambda_t}\right)^{\frac{1}{1-\alpha}}L_{d\tau}A_{d\tau}<\gamma\eta_c(1+\gamma\eta_c s_{ct})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{c\tau}}{\lambda_\tau}\right)^{\frac{1}{1-\alpha}}L_{c\tau}A_{c\tau} \tag{31}$$

that happens because of the decreasing of $\frac{\lambda_{dt}}{\lambda_t}$ in comparison to $\frac{\lambda_{ct}}{\lambda_t}$ over time because of the negative externalities carried out by the dirty input cf (47)

then if

$$\frac{\Pi_{ct}}{\Pi_{dt}}=\frac{\eta_c}{\eta_d}\left(\frac{1+\gamma\eta_c s_{ct}}{1+\gamma\eta_d s_{dt}}\right)^{-\varphi-1}\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-\varphi}(1+\tau_t)^{\varepsilon}<1$$

the social planner has to implement a subsidy q_t such that the ratio between the profit of a "clean" scientist and the profit of a "dirty" scientist will be superior to 1:

$$\frac{\Pi_{ct}}{\Pi_{dt}}=\frac{\eta_c}{\eta_d}\left(\frac{1+\gamma\eta_c s_{ct}}{1+\gamma\eta_d s_{dt}}\right)^{-\varphi-1}\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-\varphi}(1+\tau_t)^{\varepsilon}(1+q_t)>1 \tag{32}$$

and at the same time, whenever the social value of the scientist in the clean sector is equal to the social value of the worker

$$\lambda_{jt}\left(1-\alpha\right)L_{jt}^{-\alpha}\int_0^1 A_{jit}^{1-\alpha}x_{jit}^{\alpha}di=\eta_c\gamma(1+\gamma\eta_c s_{ct})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{c\tau}}{\lambda_\tau}\right)^{\frac{1}{1-\alpha}} \tag{33}$$

and if

$$\frac{\eta_c(1+\gamma)(1-\alpha)\alpha p_{ct}^{\frac{1}{1-\alpha}}L_{ct}A_{ct-1}}{(1-\alpha)\,p_{dt}L_{dt}^{-\alpha}\int_0^1 A_{dit}^{1-\alpha}x_{dit}^{\alpha}di}<>1$$

then we have to implement a subsidy or a tax (it depends) q_t' such that:

$$\frac{(1+q_t')\eta_c(1+\gamma)(1-\alpha)\alpha p_{ct}^{\frac{1}{1-\alpha}}L_{ct}A_{ct-1}}{(1-\alpha)\,L_{dt}^{-\alpha}\int_0^1 A_{dit}^{1-\alpha}x_{dit}^{\alpha}di}=1 \tag{34}$$

which yields

$$\frac{(1+q_t')\eta_c(1+\gamma)\alpha L_{ct}}{\alpha^{-\frac{1}{1-\alpha}}(1+\gamma\eta_c s_{ct})}=1$$

The appearance of $\alpha^{-\frac{1}{1-\alpha}}$ comes from the subsidy already tailored for the monopoly. Moreover, q_t' has no reason to be non permanent. Indeed, the increase of the productivity affects positively both the workers and the scientists in their activities and then incomes; there is no reason that if the ratio is equal to one at period t thanks to q_t' , it will be equal to one at period $t+1$ without q_{t+1}'.Consequently, q_t' has to be maintained over time.

Because of workers mobility accross the clean and the dirty sectors, we remind that:

$$\frac{L_{ct}}{L_{dt}} = (1+\tau_t)^{\varepsilon} \left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi} \tag{35}$$

Therefore, we have the following system:

$$\begin{aligned} &\frac{L_{ct}}{L_{dt}} = (1+\tau_t)^{\varepsilon} \left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi} \\ &\frac{(1+q_t')\eta_c(1+\gamma)\alpha L_{ct}}{\alpha^{-\frac{1}{1-\alpha}}(1+\gamma\eta_c s_{ct})} = 1 \\ &s_{ct} + L_{dt} + L_{ct} = 1 \end{aligned} \tag{36}$$

$$\lambda_{ct}\left(1-\alpha\right)L_{ct}^{-\alpha}\int_0^1 A_{cit}^{1-\alpha}x_{cit}^{\alpha}di = \gamma\eta_c(1+\gamma\eta_c s_{ct})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{c\tau}}{\lambda_\tau}\right)^{\frac{1}{1-\alpha}}L_{c\tau}A_{c\tau}$$

Then, we can deduce the exact value of s_{ct}, L_{ct}, L_{dt},q_t' at time t, but it is almost impossible analytically. Instead, in the following, we propose to find the asymptotic proportion of s_c which yields the asymptotic optimal growth rate of the economy.

Conclusion 11 *Finally, by allowing the workers to become scientists, they are able to create positive externalities. That must be enforced by a new subsidy q_t' which will be permanent. Hence, the optimal policy differs with an endogenous labor market since the social planner has to put in place another subsidy which has to be permanent. But the initial fiscal instruments remain in place. From the following part, we deduce that only the new subsidy q' and the one for the monopoly exist asymptotically. The tax τ_t and the subsidy q_t are still temporary. Then, our results matches with Stern/ Al Gore stance and are less optimistic than the results of the initial paper which yield a temporary intervention.*

3.3 The effects on the economy: environmental disaster and growth rate

This part aims at describing the asymptotic growth after redirection into the clean sector. We remind that in the initial paper, the optimal policy can only intervene on the redirection of the scientists but the government does not extract an optimal growth rate since it has to be the same as in the competitive equilibrium (equal to $\gamma\eta_{c,d} \times 1$)(because of the fixed proportion of scientists equal to 1).

We are able to extract the optimal asymptotic proportion of scientists of the clean sector thanks to the equalization of the social value of a worker and the social value of a scientist working in the clean sector. We have:

$$\lambda_{ct}\,(1-\alpha)\,L_{ct}^{-\alpha}\int_0^1 A_{cit}^{1-\alpha}x_{cit}^{\alpha}di = \gamma\eta_c(1+\gamma\eta_c s_{ct})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{c\tau}}{\lambda_\tau}\right)^{\frac{1}{1-\alpha}}L_{c\tau}A_{c\tau}$$

We can extract the asymptotic value of s_c and more generally the asymptotic composition of the labour market, which yields the economic growth.

PS: in order to ease the computations, we use the following specification (with $\sigma = 2$,we match with Nordhaus specification):

$$u(C_t, S_t) = \frac{C_t^{1-\sigma}}{1-\sigma} + \nu(S_t)$$

if $\gamma\eta_c > \rho((1-\alpha)\,(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}} - \gamma\eta_c)$

$$s_c = \frac{\gamma\eta_c(1+\rho) - (1-\alpha)\,(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}}\rho}{(\gamma\eta_c(1+\rho)\,(1-\alpha)\,(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}}) + \gamma\eta_c(1+\rho)} \tag{37}$$

$$L_c = \frac{\gamma\eta_c(1+\rho)\,(1-\alpha)\,(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}} + (1-\alpha)\,(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}}\rho}{(\gamma\eta_c(1+\rho)\,(1-\alpha)\,(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}}) + \gamma\eta_c(1+\rho)}$$

Hence, thanks to (36), we will find the asymptotic optimal subsidy q' thanks to the equalization of the profits between the workers in the clean sector and the scientists in the clean sector, asymptotically we have:

$$\frac{(1+q')\eta_c(1+\gamma)\alpha(1-s_c)}{\alpha^{-\frac{1}{1-\alpha}}(1+\gamma\eta_c s_c)} = 1$$

using (37), we deduce q' the asymptotic optimal subsidy in function of $\eta_c, \gamma, \alpha\,, g, \rho$.

$$q' = \frac{\alpha^{-\frac{1}{1-\alpha}}(1+\gamma\eta_c\frac{\gamma\eta_c(1+\rho)-(1-\alpha)(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}}\rho}{(\gamma\eta_c(1+\rho)(1-\alpha)(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}})+\gamma\eta_c(1+\rho)}) - \eta_c(1+\gamma)\alpha(1-\frac{\gamma\eta_c(1+\rho)-(1-\alpha)(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}}\rho}{(\gamma\eta_c(1+\rho)(1-\alpha)(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}})+\gamma\eta_c(1+\rho)})}{\eta_c(1+\gamma)\alpha(1-\frac{\gamma\eta_c(1+\rho)-(1-\alpha)(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}}\rho}{(\gamma\eta_c(1+\rho)(1-\alpha)(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}})+\gamma\eta_c(1+\rho)})}$$

q' is decreasing in ρ which means that the lower our consideration to the future generations is (ρ high), the lower is the subsidy.

if $\gamma\eta_c \leq \rho((1-\alpha)\left(\frac{1}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} - \gamma\eta_c)$

$$s_c = 0$$

$$L_c = 1$$

Thanks to (17), we have the asymptotical growth rate of the dirty input and of the economy

if $\gamma\eta_c > \rho((1-\alpha)\left(\frac{1}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} - \gamma\eta_c)$

$$g = \frac{\gamma\eta_c(1+\rho) - (1-\alpha)\left(\frac{1}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}\rho}{(1+\rho)(1-\alpha)\left(\frac{1}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} + 1 + \rho}$$

if $\gamma\eta_c \leq \rho((1-\alpha)\left(\frac{1}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} - \gamma\eta_c)(*)$

$$g = 0$$

Hence, as in the competitive equilibrium the proportion of scientists may be equal to 0, consequently so does for the economic growth. In fact, the condition $\gamma\eta_c \leq \rho((1-\alpha)\left(\frac{1}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} - \gamma\eta_c)$ means that if ρ is too high in comparison to the other parameters, there is no growth. In that case, when ρ verifies (*), we do not care sufficiently about future generations for allowing them to experience economic growth. The asymptotic consumption will be constant, the asymptotic value of the environment will be equal to $\overline{S}$ thanks to the switch towards the clean sector and the regeneration scheme, hence the utility is constant $u(\overline{C}, \overline{S})$, as in the competitive equilibrium. Therefore, the intervention of the government is "*asymptotically useless*" except for correcting the monopoly problem in the intermediary machines and redirecting temporary towards the clean sector. With the asymptotical constant consumption and utility, we retrieve the results of the Hartwick's rule and the Maximin principle without using resources and the Hotelling rule. In our framework, the constant consumption comes from the endogenous labour market and the endogenous growth scheme and not from the use of ressources and the Hotelling rule.

When (*) is not verified then, s_c is a decreasing function of ρ, that means, the higher is our consideration for the future generations, the higher is the asymptotic growth rate. s_c is increasing in γ and η_c that means the higher is the productivity and the probability of enhancing the productivity, the higher is the asymptotic optimal growth. Lastly, s_c is also increasing in α. Hence, s_c is affected by the same variables and in the same way as in the competitive equilibrium except the discount factor ρ which plays a decisive role. L_c is always positive, indeed we remind that L_c is a *necessary* factor for producing Y_c. In fact, with another framework, we retrieve the usual result of the macroecomics model with micro foundations: the asymptotic consumption growth is decreasing in ρ.

Moreover, when $\gamma\eta_c > \rho((1-\alpha)\,(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}} - \gamma\eta_c)$ and $\alpha\eta_d(1+\gamma) > 1$, we can compare the asymptotic growth rates in equilibrium and in the optimal policy, which are both positives. We find ρ^* a critical threshold such that if $\rho > \rho^*$ then $g_{optpolicy} < g_{coequilibrium}$.

Conclusion 12 *As in the competitive equilibrium, there is a case when there is no growth. The asymptotic optimal growth can be equal to 0 if ρ is sufficiently high. Then, it is possible to find the exact value of consumption and the constant utility. Moreover, we find that there is another threshold ρ^*which with the asymptotic growth in competitive equilibrium is higher than the asymptotic optimal growth. Concerning the asymptotic growth rates, there are huge differences between the initial model and our model. In our model, through the endogenous labour market, ρ plays a negative role on the asymptotic optimal growth, whereas in the initial paper the asymptotic optimal growth is set to $\gamma\eta_c$ even after redirection and optimal policy. This happens because of the fixed proportion of scientists to 1. In our framework, the growth rates can be different both in the competitive equilibrium and in the optimal policy.*

4 Further ideas

4.1 Introducing demographic growth

We could introduce demographic growth, exogenous as Stiglitz did in 1974. The demographic growth is legitimate since we solve the problem asymptotically and we dealth with long term changes suchs the climate change and technological progress. However, it causes many problems within the framework. In order to avoid problem of sharing profits of the monopoly, the number of machines has to increase as the same growth rate as the population. We denote P_t as the total population composed by scientists s_t and workers L_t :

$$P_{t+1} = P_t(1+n)$$
$$\left(\int_0^1 x_{cit+1}di + \int_0^1 x_{dit+1}di\right) = (1+n)\left(\int_0^1 x_{cit}di + \int_0^1 x_{dit}di\right)$$

The other equations of the framework remain identical. The solution of the competitive problem will not be static but dynamic since the above equations are taken into account in the program of maximization of the producers. We could have some insights of what could happen. Since the demographic rate is equal to n,the endogenous technological progress will be higher, then if it is not redirected the environmental disaster will be reach sooner. Moreover, we do feel that the demographic growth rate could play an interesting role in the allocations of scientists and workers in the economy, n could interfere in the allocations.

If we introduce an endogenous demographic growth, we could think that the higher is the quality of the environment, the higher is the demographic growth. However, we do believe that the low quality of the environment could encourage higher fertility rates since the children can be seen as insurance in a dangerous environment (see Equatorial Africa for instance). Then, further studies have to conducted and empirical works are necessary in order to estimate the relationship existing between the environment and the fertility rate. (see Geography, Demography and Economic Growth in Africa, J.Sachs & D.Bloom, 1998).

4.2 Education

At first sight, we wanted to introduce an education cost c standing for the wish to become a scientist. Consequently, this separable cost could apply only for scientists, since becoming scientist demands more education. However, the individuals live for one period and they choose their job at the beginning of the period. In fact, they do not live for time $t > 1$ and they do not have the opportunity to change their work. Introducing the education cost c seems irrelevant in our framework. However, if the individuals lived for more than one period, they could change their job. It could be interesting to introduce such a cost since it would be another variable directing the individuals. The choice of an individual would be:

if $w_t + c < \pi_t$, the individual would become a scientist, either a worker.

However, by introducing a " dynamic " into the choices of the individual, we have to decide:

- how many periods they live
- if they can change their job.

Hence, we would expect a proportion of scientists and the growth rate depending on the education cost. However, it seems difficult to introduce overlapping-generations scheme in this framework.

4.3 Quantitative example : simulation

Based on the Stern and Nordhaus results, the authors proposed a quantitative example of the optimal policy. We find that the specification of the parameters has to be different in order to match with the Nordhaus results since our asymptotic optimal growth rate is different. Moreover, we find that taking into account the initial calibrations, there is never growth in the competitive equilibrium because of the condition on the expected productivity.

A Appendix: proofs of the competitive equilibrium

We choose to demonstrate the results of the initial paper in order to show as much as possible the occurring mechanisms of the model.

To find the relative price, we maximize

$$\underset{Y_{dt},Y_{ct}}{Max} = Y_t - p_{ct}Y_{ct} - p_{dt}Y_{dt}$$

The CPO yields (14)

We then get the demand of machines i in sector j thanks to the profit maximization of the producer of the input Y_{jt} :

$$\underset{L_{jt},x_{jit},i\in[0,1]}{Max} p_{jt}L_{jt}^{1-\alpha}\int_0^1 A_{jit}^{1-\alpha}x_{jit}^{\alpha}di - w_tL_t - \int_0^1 p_{jit}x_{jit}di$$

Taking the CPO with each x_{jit}, we deduce the demand of each machine i, in sector j,

$$x_{jit} = \left(\frac{p_{jit}}{p_{jt}}\right)^{\frac{1}{1-\alpha}} L_{jt}A_{jit}$$

Taking the CPO with L_{jt} yields the wage of a worker in both sectors:

$$w_t = p_{jt}(1-\alpha)L_{jt}^{-\alpha}\int_0^1 A_{jit}^{1-\alpha}x_{jit}^{\alpha}di \tag{38}$$

As for the monopolist of each machine i in sector j, we solve the problem:

$$\underset{p_{jit},x_{jit}}{Max}\,(p_{jit}-\alpha^2)x_{jit}$$

and we derive the equilibrium demand for machine i in sector j thanks to the mark-up formula and the iso elastic demand.

$$x_{jit} = p_{jt}^{\frac{1}{1-\alpha}}L_{jt}A_{jit} \tag{39}$$

hence, using (13), and the previous results, we deduce

$$\frac{p_{ct}}{p_{dt}} = \left(\frac{A_{ct}}{A_{dt}}\right)^{-(1-\alpha)} \tag{40}$$

and with respect to L_{ct} and L_{dt} , we obtain

$$\frac{L_{ct}}{L_{dt}} = \left(\frac{p_{ct}}{p_{dt}}\right)^{-\frac{\varphi-1}{1-\alpha}}\frac{A_{dt}}{A_{ct}} = \left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi} \tag{41}$$

Complete proof of (19), (20), (21).

In order to resolve the system of three equations with three unknows, we use (39) for replacing x_{dit} and (40)

$$\eta_d(1+\gamma)(1-\alpha)\alpha p_{dt}^{\frac{1}{1-\alpha}} L_{dt} A_{dt-1} = (1-\alpha)\, p_{dt} L_{dt}^{-\alpha} \int_0^1 A_{dit}^{1-\alpha} x_{dit}^{\alpha} di$$

becomes

$$\begin{aligned}
\eta_d(1+\gamma)\alpha p_{dt}^{\frac{1}{1-\alpha}} L_{dt} A_{dt-1} &= p_{dt} L_{dt}^{-\alpha} \int_0^1 A_{dit}^{1-\alpha} \left(p_{dt}^{\frac{1}{1-\alpha}} L_{dt} A_{dit} \right)^{\alpha} di \\
\eta_d(1+\gamma)\alpha p_{dt}^{\frac{1}{1-\alpha}} L_{dt} A_{dt-1} &= (p_{dt})^{\frac{1}{1-\alpha}} \int_0^1 A_{dit} di \\
\eta_d(1+\gamma)\alpha L_{dt} A_{dt-1} &= A_{dt} \\
\eta_d(1+\gamma)\alpha L_{dt} &= (1+\gamma\eta_d s_{dt}) \\
L_{dt} &= \frac{1+\gamma\eta_d s_{dt}}{\eta_d(1+\gamma)\alpha}
\end{aligned}$$

Taking (41) we express L_{ct},

$$L_{ct} = \left(\frac{A_{ct}}{A_{dt}} \right)^{-\varphi} \left(\frac{1+\gamma\eta_d s_{dt}}{\eta_d(1+\gamma)\alpha} \right)$$

and using the market clearing condition, we deduce s_{dt} and hence L_{ct} and L_{dt}.

When there is no growth in the economy, to prove (23), we have an arithmetic-geometrical progression:

$$S_{t+1} = -\xi Y_{do} + (1+\delta) S_t$$

we find the fixed point $l = \xi \frac{Y_{do}}{\delta}$.We deduce (23) ($Y_{do}$ is the constant dirty input).

B Appendix: proofs of the optimal policy

The new program is

$$\underset{Y_t, Y_{ct}, Y_{dt}, C_t, S_t, x_{dit}, x_{cit}, L_{dt}, L_{ct}, s_{ct}, s_{dt}, A_{dt}, A_{ct}, A_{cit,} A_{dit}, L_{t,} s_t}{Max} \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} u(C_t, S_{t)}$$

under the constraints (4),(5),(8),(9),(11),(12),(13) and the new constraints (27),(28),(29):

We remind that the Lagrangean multiplier represents the shadow price or the social value of the variable.

with λ_t the lagragean multiplier for (4), the FOC with respect to Y_t, makes equal the lagrangean multiplier χ_t of (8) with λ_t.

Hence, the FOC with respect to C_t yields:

$$\lambda_t = \frac{1}{(1+\rho)^t} \frac{\partial u\ (C_t, S_t)}{\partial C_t} = \chi_t \tag{42}$$

We call ω_t the lagrangean multplier for (13), the evolution of the environmental stock. The FOC with respect to S_t yields:

$$\omega_t = \frac{1}{(1+\rho)^t} \frac{\partial u(C_t, S_t)}{\partial S_t} + (1+\delta) 1_{S_t < \overline{S}} \omega_{t+1}$$

which derives from the fact that the environmental stock evolves and S_{t+1} is in the equation traducing the evolution of the stock. $1_{S_t<\overline{S}}$ is equal to 1 if $S_t < \overline{S}$ either 0

and using the operator forward F, we have

$$\begin{aligned} \omega_t (1 - (1+\delta) 1_{S_t < \overline{S}} F) &= \frac{1}{(1+\rho)^t} \frac{\partial u(C_t, S_t)}{\partial S_t} \\ \omega_t &= \sum_{v=t}^{\infty} (1+\delta)^{v-t} \frac{1}{(1+\rho)^t} \frac{\partial u(C_v, S_v)}{\partial S_v} 1_{S_t, \ldots, S_v < \overline{S}} \end{aligned} \tag{43}$$

We derive the lagrangean with respect to x_{jit}, and we obtain the necessary subvention in order to correct for the monopoly, the demand for x_{jit} changes in comparison to the competitive equilibrium:

$$x_{jit} = \left(\frac{1}{\alpha} \widehat{p_{jt}} \right)^{\frac{1}{1-\alpha}} A_{jit} L_{jt} \tag{44}$$

$$Y_{jt} = \left(\frac{1}{\alpha} \widehat{p_{jt}} \right)^{\frac{\alpha}{1-\alpha}} A_{jt} L_{jt} \tag{45}$$

with $\widehat{p_{jt}} = \frac{\lambda_{jt}}{\lambda_t}$, which are the shadow prices of the input Y_{jt} relatively to the price of the final good Y_t. Moreover, λ_{dt} and λ_{ct} are the lagrangeans for (5) and (6), the FOC with respect to Y_{dt} and Y_{ct}, we have:

$$Y_{ct}^{-\frac{1}{\varepsilon}} (Y_{ct}^{\frac{\varepsilon-1}{\varepsilon}} + Y_{dt}^{\frac{\varepsilon-1}{\varepsilon}})^{\frac{1}{\varepsilon-1}} = \frac{\lambda_{ct}}{\lambda_t} \tag{46}$$

$$Y_{dt}^{-\frac{1}{\varepsilon}} (Y_{ct}^{\frac{\varepsilon-1}{\varepsilon}} + Y_{dt}^{\frac{\varepsilon-1}{\varepsilon}})^{\frac{1}{\varepsilon-1}} - \frac{\omega_{t+1} \xi}{\lambda_t} = \frac{\lambda_{dt}}{\lambda_t} \tag{47}$$

There is a distortion in comparison to the competitive equilibrium because of the externality produced by the input of the dirty sector Y_{dt}.

Consequently, we set a tax equal to

$$\tau_t = \frac{\omega_{t+1}\xi}{\lambda_{dt}} \tag{48}$$

and we deduce the value of the tax thanks to (43)

τ_t has an impact on L_{ct} and L_{dt} :

$$\frac{L_{ct}}{L_{dt}} = (1+\tau_t)^{\varepsilon}\left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi} \tag{49}$$

We denote μ_{dt} and μ_{ct} as the lagrangean multipliers for (12), the FOC with respect to A_{dt} and A_{ct} yield for $j = c, d$:

$$\mu_{jt} = \lambda_t \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} (1-\alpha)\left(\frac{\lambda_{jt}}{\lambda_t}\right)^{\frac{1}{1-\alpha}} L_{jt} + (1+\gamma\eta_j s_{jt+1})\mu_{jt+1}$$

because of the endogenous growth, the shadow price or the social value of the productivity is represented by a suite.

Thanks to the operator forward F and (12) we obtain:

$$\mu_{jt} = (1+\gamma\eta_j s_{jt})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{jt}}{\lambda_t}\right)^{\frac{1}{1-\alpha}} L_{j\tau}A_{j\tau} \tag{50}$$

Because of endogenous growth, the shadow price of productivity captures the fact that the present productivity will have future effects for every $\tau > t$, hence the shadow price is expressed as a series.

In order to control the productivity, the social planner can redirect the scientist across the sectors. We denote β_t the lagrangean multiplier for the new constraint (28) and ζ_t for (29) , and ϕ_c,ϕ_d for the slackness conditions. If we derive, the lagrangean with respect to L_t,

we get

$$\beta_t = \zeta_t$$

and κ_t the lagrangean multiplier for (27) the FOC with respect to s_t,

$$\kappa_t = \zeta_t$$

hence

$$\kappa_t = \beta_t$$

which means that the optimal allocations of ressources is reached when the social value of a scientist is equal to the social value of a worker.

The FOC with respect to L_{ct} and L_{dt}, yields :

$$\lambda_{ct}\left(1-\alpha\right)L_{ct}^{-\alpha}\int_0^1 A_{cit}^{1-\alpha}x_{cit}^{\alpha}di=-\beta_t=-\kappa_t \tag{51}$$

$$\lambda_{dt}\left(1-\alpha\right)L_{dt}^{-\alpha}\int_0^1 A_{dit}^{1-\alpha}x_{dit}^{\alpha}di=-\beta_t=-\kappa_t \tag{52}$$

Hence, at the optimum, the social value of a worker in the dirty sector has to be equal to the social value of a worker in the clean sector, which is exactly what happens in the competitive equilibrium due to the equalization of wages.

Then, if we derive the lagrangean with respect to s_{ct} and s_{dt},we obtain:

$$\mu_{dt}\eta_d\gamma=-\kappa_t-\phi_d$$
$$\mu_{ct}\eta_c\gamma=-\kappa_t-\phi_c$$

4 different cases because of the slackness conditions:

- if s_{dt} >0 and s_{ct} >0, $\phi_c=\phi_d=0$

The social value of each marginal scientist in sector j, is given by $\mu_{jt}\gamma\eta_j$,hence the social planner will allocate the scientist according to the following ratio:

$$\mu_{dt}\eta_d\gamma=-\kappa_t=\mu_{ct}\eta_c\gamma$$

$$\gamma\eta_c(1+\gamma\eta_c s_{ct})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{ct}}{\lambda_t}\right)^{\frac{1}{1-\alpha}}L_{c\tau}A_{c\tau}=\gamma\eta_d(1+\gamma\eta_d s_{dt})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{dt}}{\lambda_t}\right)^{\frac{1}{1-\alpha}}L_{d\tau}A_{d\tau}$$

and given (52), (51), we also have

$$\lambda_{jt}\left(1-\alpha\right)L_{jt}^{-\alpha}\int_0^1 A_{jit}^{1-\alpha}x_{jit}^{\alpha}di=\eta_l\gamma(1+\gamma\eta_l s_{lt})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{lt}}{\lambda_t}\right)^{\frac{1}{1-\alpha}}L_{l\tau}A_{l\tau},\text{for } j=c,d \text{ and } l=c,d$$

- if $s_{dt}>0$ and $s_{ct}=0,\phi_d=0,\phi_c>0$

The FOC with respect to s_{ct} and s_{dt} become

$$\begin{aligned}\mu_{dt}\gamma\eta_d &= -\kappa_t\\ \mu_{ct}\gamma\eta_c &= -\kappa_t-\phi_c\\ \mu_{dt}\gamma\eta_d &> \mu_{ct}\gamma\eta_c\end{aligned}$$

hence,

$$\gamma\eta_d(1+\gamma\eta_d s_{dt})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{dt}}{\lambda_t}\right)^{\frac{1}{1-\alpha}}L_{d\tau}A_{d\tau}>\gamma\eta_c(1+\gamma\eta_c s_{ct})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{ct}}{\lambda_t}\right)^{\frac{1}{1-\alpha}}L_{c\tau}A_{c\tau}$$

and given (52), (51)

$$\lambda_{jt}\left(1-\alpha\right)L_{jt}^{-\alpha}\int_0^1 A_{jit}^{1-\alpha}x_{jit}^{\alpha}di > \gamma\eta_c(1+\gamma\eta_c s_{ct})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{ct}}{\lambda_t}\right)^{\frac{1}{1-\alpha}}L_{c\tau}A_{c\tau},\ \text{for } j=c,d$$

- if $s_{ct}>0$ and $s_{dt}=0, \phi_c=0, \phi_d>0$

The FOC with respect to s_{ct} and s_{dt} become

$$\begin{aligned}
\mu_{ct}\gamma\eta_c &= -\kappa_t \\
\mu_{dt}\gamma\eta_d &= -\kappa_t-\phi_d \\
\mu_{ct}\gamma\eta_c &> \mu_{dt}\gamma\eta_d
\end{aligned}$$

hence

$$\gamma\eta_c(1+\gamma\eta_c s_{ct})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{ct}}{\lambda_t}\right)^{\frac{1}{1-\alpha}}L_{c\tau}A_{c\tau} > \gamma\eta_d(1+\gamma\eta_d s_{dt})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{d\tau}}{\lambda_\tau}\right)^{\frac{1}{1-\alpha}}L_{d\tau}A_{d\tau}$$

and given (52), (51)

$$\lambda_{jt}\left(1-\alpha\right)L_{jt}^{-\alpha}\int_0^1 A_{jit}^{1-\alpha}x_{jit}^{\alpha}di > \gamma\eta_d(1+\gamma\eta_d s_{dt})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{d\tau}}{\lambda_\tau}\right)^{\frac{1}{1-\alpha}}L_{d\tau}A_{d\tau},\ \text{for } j=c,d$$

- if $s_{ct}=0$ and $s_{dt}=0, \phi_c>0, \phi_d>0$

that means there is no growth.
The FOC with respect to s_{ct} and s_{dt} become

$$\begin{aligned}
\mu_{dt}\gamma\eta_d &= -\kappa_t-\phi_d \\
\mu_{ct}\gamma\eta_c &= -\kappa_t-\phi_c \\
\mu_{ct}\gamma\eta_c+\phi_c &= \mu_{dt}\gamma\eta_d+\phi_d
\end{aligned}$$

Given these orders of social values, the social planner will redirect the workers and the scientists across the sector *and* across the jobs thanks to subsidies in order to make the profits matching with the order set by the social values.

According to the computations,

$$\gamma\eta_d(1+\gamma\eta_d s_{dt})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{dt}}{\lambda_t}\right)^{\frac{1}{1-\alpha}}L_{d\tau}A_{d\tau} < \gamma\eta_c(1+\gamma\eta_c s_{ct})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{c\tau}}{\lambda_\tau}\right)^{\frac{1}{1-\alpha}}L_{c\tau}A_{c\tau}$$

Hence, we go back to the profit ratio,
if

$$\frac{\Pi_{ct}}{\Pi_{dt}}=\frac{\eta_c}{\eta_d}\left(\frac{1+\gamma\eta_c s_{ct}}{1+\gamma\eta_d s_{dt}}\right)^{-\varphi-1}\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-\varphi}(1+\tau_t)^{\varepsilon}<1$$

then **we have to implement a subsidy** q_t such that:

$$\frac{\Pi_{ct}}{\Pi_{dt}}=\frac{\eta_c}{\eta_d}\left(\frac{1+\gamma\eta_c s_{ct}}{1+\gamma\eta_d s_{dt}}\right)^{-\varphi-1}\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-\varphi}(1+\tau_t)^{\varepsilon}(1+q_t)>1$$

we then have $s_{ct}>0$, and $s_{dt}=0$
and given (52), (51) if:

$$\lambda_{jt}\left(1-\alpha\right)L_{jt}^{-\alpha}\int_0^1 A_{jit}^{1-\alpha}x_{jit}^{\alpha}di=\gamma\eta_c(1+\gamma\eta_c s_{ct})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{c\tau}}{\lambda_\tau}\right)^{\frac{1}{1-\alpha}}L_{c\tau}A_{c\tau},\ \text{for } j=c,d$$

hence the social value of a scientist in the clean sector is superior to the one's in the dirty sector and is superior to the social value of a worker in both sector
if

$$\frac{\eta_c(1+\gamma)(1-\alpha)\alpha p_{ct}^{\frac{1}{1-\alpha}}L_{ct}A_{ct-1}}{\left(1-\alpha\right)p_{dt}L_{dt}^{-\alpha}\int_0^1 A_{dit}^{1-\alpha}x_{dit}^{\alpha}di}<>1$$

then **we have to implement a subsidy** q_t' such that:
thanks to (40), (41):

$$\frac{(1+q_t')\eta_c(1+\gamma)(1-\alpha)\alpha p_{ct}^{\frac{1}{1-\alpha}}L_{ct}A_{ct-1}}{\left(1-\alpha\right)p_{dt}L_{dt}^{-\alpha}\int_0^1 A_{dit}^{1-\alpha}x_{dit}^{\alpha}di}=1$$

$$\frac{(1+q_t')\eta_c(1+\gamma)\alpha A_{dt}L_{ct}A_{ct-1}}{\alpha^{-\frac{1}{1-\alpha}}A_{ct}\int_0^1 A_{dit}di}=1$$

$$\frac{(1+q_t')\eta_c(1+\gamma)\alpha L_{ct}}{\alpha^{-\frac{1}{1-\alpha}}(1+\gamma\eta_c s_{ct})}=1$$

We demonstrate how we find the asymptotical proportion of scientists after the redirection towards the clean sector (s_{dt} is already equal to 0):

First we have

$$\begin{aligned}\frac{L_{ct}}{L_{dt}} &= (1+\tau_t)^{\varepsilon}\left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi}\\ L_{ct}+L_{dt}+s_{ct} &= 1\end{aligned}$$

by taking the limit we deduce

$$\begin{aligned}L_c+s_s &= 1\\ L_d &= 0\end{aligned}\tag{53}$$

Second, from the maximization, we equalize the social value of a worker of a clean sector and the social value of a scientist working in the clean sector.

$$\lambda_{ct}\,(1-\alpha)\,L_{ct}^{-\alpha}\int_0^1 A_{cit}^{1-\alpha}x_{cit}^{\alpha}di = \gamma\eta_c(1+\gamma\eta_c s_{ct})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{c\tau}}{\lambda_\tau}\right)^{\frac{1}{1-\alpha}}L_{c\tau}A_{c\tau}$$

by using (44)

$$\lambda_{ct}\,(1-\alpha)\,(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}}\left(\frac{\lambda_{ct}}{\lambda_t}\right)^{\frac{\alpha}{1-\alpha}}A_{ct} = \gamma\eta_c(1+\gamma\eta_c s_{ct})^{-1}\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{c\tau}}{\lambda_\tau}\right)^{\frac{1}{1-\alpha}}L_{c\tau}A_{c\tau}$$

ordering and using (12) τ times

$$\lambda_t(1+\gamma\eta_c s_{ct})\,(1-\alpha)\,(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}}\left(\frac{\lambda_{ct}}{\lambda_t}\right)^{\frac{1}{1-\alpha}}A_{ct} = \gamma\eta_c\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{c\tau}}{\lambda_\tau}\right)^{\frac{1}{1-\alpha}}L_{c\tau}A_{ct}(1+\gamma\eta_c s_{c\tau})^{\tau-t}$$

ordering, taking the limits for $L_{c\tau}$, s_{ct}, $s_{c\tau}$ and using (53)

$$\lambda_t(1+\gamma\eta_c s_c)\,(1-\alpha)\,(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}}\left(\frac{\lambda_{ct}}{\lambda_t}\right)^{\frac{1}{1-\alpha}}A_{ct} = \gamma\eta_c\sum_{\tau\geq t}\lambda_\tau\left(\frac{\lambda_{c\tau}}{\lambda_\tau}\right)^{\frac{1}{1-\alpha}}L_{c\tau}A_{ct}(1+\gamma\eta_c s_{c\tau})^{\tau-t}$$

then using (42)

$$(1+\gamma\eta_c s_c)\,(1-\alpha)\,(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}} = \gamma\eta_c(1-s_c)\sum_{\tau\geq t}\frac{1}{(1+\rho)^{\tau-t}}(1+\gamma\eta_c s_c)^{\tau-t}\frac{\frac{\partial u\ (C_\tau,\overline{S})}{\partial C_\tau}}{\frac{\partial u\ (C_t,\overline{S})}{\partial C_t}}$$

then using what follows

$$(1+\gamma\eta_c s_c)\,(1-\alpha)\,(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}} = \gamma\eta_c(1-s_c)\sum_{\tau\geq t}\frac{1}{(1+\rho)^{\tau-t}}\times\frac{1}{(1+\gamma\eta_c s_c)^{\tau-t}}$$

using the formula of a geometrical series

$$(1+\gamma\eta_c s_c)\,(1-\alpha)\,(\frac{1}{\alpha})^{\frac{\alpha}{1-\alpha}} = \gamma\eta_c(1-s_c)\frac{1}{1-\frac{1}{(1+\rho)}\times\frac{1}{(1+\gamma\eta_c s_c)}}$$

hence we deduce (37)

PS: since there is no dirty sector left asymptotically: $\lim\frac{\lambda_{ct}}{\lambda_t}=1$.

Furthermore, asymptotically if the technical change has been redirected towards the clean sector, $S_t=\overline{S}$.

Moreover thanks to (8), (44), (17):

$$\begin{aligned}
\frac{\partial u\ (C_t,\overline{S})}{\partial C_t} &= (Y_t - \alpha^2 \int_0^1 x_{cit} di)^{-2} \\
&= ((A_{ct}^{\varphi} + A_{dt}^{\varphi})^{-\frac{1}{\varphi}} A_{ct}A_{dt} - \alpha^2 \int_0^1 \left(\frac{1}{\alpha}\widehat{p_{ct}}\right)^{\frac{1}{1-\alpha}} A_{cit}L_{ct}di)^{-2} \\
&= (A_{ct}^{\varphi} + A_{dt}^{\varphi})^{-\frac{1}{\varphi}} A_{ct}A_{dt} - \alpha^2 \left(\frac{1}{\alpha}\widehat{p_{ct}}\right)^{\frac{1}{1-\alpha}} A_{ct}L_{ct})^{-2}
\end{aligned}$$

taking the limit we have an equivalent

$$\alpha^4 \left(\frac{1}{\alpha}\right)^{\frac{2}{1-\alpha}} {A_{ct}}^2 (1-s_c)^2$$

$\dfrac{\partial u\ (C_\tau,\overline{S})}{\partial C_\tau}$ is equivalent to $\alpha^4 \left(\frac{1}{\alpha}\right)^{\frac{2}{1-\alpha}} {A_{ct}}^2 (1-s_c)^2(1+\gamma\eta_c s_c)^{2(\tau-t)}$

We mention the main papers used to build the initial framework and our framework

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